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Vertex operators and symmetric functions

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Abstract. An algorithm for the calculation of q-dependent spin characters of the symmetric group is given with an explicit example of S_4. A method for constructing the q-analogue of the vertex operators is developed. A 1:1 correspondence between the space $\mathcal{V}$ of twisted q-vertex operators and the ring of q-deformed symmetric functions $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}(q,t)$ is established and a mapping from $\mathcal{V} \to \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}(q,t)$ is defined. A number of relevant theorems are given.

1. Introduction

The development of methods for constructing and studying integrable quantum models has recently led to new algebraic structures known as quantum groups [1] or, more precisely, quantum affine Lie algebras. Finding vertex operator representations of quantum affine algebras is a natural issue in the study of quantum groups. Besides, recent progress in conformal field theories has shown the important role played by vertex operator algebras in quantum field theories [4].

These developments have stimulated much activity in both mathematicians and physicists. In a recent paper [2] Frenkel and Jing have constructed the untwisted vertex representations of quantum affine algebras and more recently Jing [7] has developed the twisted q-vertex operators. Drinfeld's theorem of quantum affine algebras [1] plays the crucial role in such constructions.

First of all we will reconstruct the ring $\Lambda_{\mathbb{Q}}^{q}$ of q-deformed symmetric functions by using a different type of q-deformation then we will show that there exists an isomorphism between the ring $\Lambda_{\mathbb{Q}}^{q}$ and the space $\mathcal{V}_q$ of q-deformed vertex operators. These q-deformed vertex operators are nothing but the q-analogue of the untwisted vertex operators used in the description of affine Kac–Moody algebras [4]. This leads to a very simple way of constructing the twisted q-vertex operators.

In this paper we will closely follow the notation of [9, 10, 13] and will use the results therein.

2. The ring $\Lambda_{\mathbb{Q}}^{q}$

In [13], we gave the q-deformation of the Hall–Littlewood symmetric function $P_\lambda(s,t)$ using the following definition of q-number

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}. \tag{1}$$

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It is possible to consistently define various types of q-deformations of symmetric functions such as in terms of the q-numbers

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{2}$$

as used in the description of quantum groups [1]. We will use (2) for the definition of a q-number unless specified otherwise. The q-analogue of the Hall–Littlewood symmetric functions will form the basis of the ring Λ_Q^q of the q-deformed symmetric functions. A q-analogue of complete symmetric functions can be defined as

$$h_\lambda^q = \sum_{|\lambda|=n} (z_\lambda^q)^{-1} p_\lambda^q \tag{3}$$

where

$$z_\lambda^q = \prod [i]_q^{m_i} [m_i]_q!. \tag{4}$$

In [10] it has been shown that $P_\lambda(s,t)$ is the generalized form of the Hall–Littlewood symmetric function. Let us define a scalar product $\langle\ ,\ \rangle_{(s,t)}^{(q)}$ over $Q_q(s,t)$ as follows

$$\langle p_\lambda, p_\mu \rangle_{(s,t)}^{(q)} = \delta_{\lambda\mu} z_\lambda^q(s,t) \tag{5}$$

where

$$z_\lambda^q(s,t) = \prod_i [i]_q^{m_i} [m_i]_q! \prod_j^{l(\lambda)} \frac{(1 - s^{[\lambda_j]_q})}{(1 - t^{[\lambda_j]_q})} \tag{6}$$

and $Q_q(s,t)$ is the q-analogue of the field of rational functions in independent indeterminates s and t. We call $P_\lambda^q(s,t)$, the q-deformation of the symmetric function $P_\lambda(s,t)$ and define

$$\mathcal{P}_q = \prod_{i,j} \left\{ \frac{(tx_i y_j; s)_\infty}{(x_i y_j; s)_\infty} \right\}_q$$

where

$$(a; s)_\infty = \prod_{r=0}^{\infty} (1 - as^r)$$

and the subscript q in $\{\ \}$ indicates that the powers of t and s are q-numbers.

$$\mathcal{P}_q(x, y; s, t) = \sum_\lambda z_\lambda^q(s,t)^{-1} p_\lambda(x) p_\lambda(y). \tag{7}$$

Proof. We compute $\exp(\log \mathcal{P}_q)$;

$$\log \mathcal{P}_q = \sum_{i,j}\sum_{r=0}^{\infty}\left\{\log(1-x_iy_js^r)^{-1} - \log(1-tx_iy_js^r)^{-1}\right\}_q$$

$$= \sum_{i,j}\sum_{r=0}^{\infty}\sum_{n\geqslant 1}\frac{1}{[n]_q}(x_iy_js^r)^{[n]_q}\left(1-t^{[n]_q}\right)$$

$$= \sum_{n\geqslant 1}\frac{1}{[n]_q}\frac{(1-t^{[n]_q})}{(1-s^{[n]_q})}p_n(x)p_n(y).$$

Hence

$$\mathcal{P}_q = \prod_{n\geqslant 1}\exp\left(\frac{1}{[n]_q}\frac{(1-t^{[n]_q})}{(1-s^{[n]_q})}p_n(x)p_n(y)\right)$$

$$= \prod_{n\geqslant 1}\sum_{m_n=1}^{\infty}\frac{1}{[m_n]_q!}\left(\frac{1}{[n]_q}\frac{(1-t^{[n]_q})}{(1-s^{[n]_q})}p_n(x)p_n(y)\right)^{m_n}$$

in which the coefficient of $p_\lambda(x)p_\lambda(y)$ is seen to be $z_\lambda^q(s,t)^{-1}$. Here we have made use of the q-exponential function defined as

$$e_q^x = \sum_{n\geqslant 1}^{\infty}\frac{x^n}{[n]_q!}.$$

Hence for $s=0$ we get

$$\mathcal{P}_q = \sum_\lambda b_\lambda^q(t)P_\lambda^q(x;t)P_\lambda^q(y;t)$$

where $P_\lambda^q(x;t)$ and $P_\lambda^q(y;t)$ are the q-deformed Hall–Littlewood symmetric functions and will be denoted by $P_\lambda^q(t)$, and $b_\lambda^q(t)$ is defined in (11).

Expression (7) is a very general definition of symmetric functions and all the symmetric functions (Hall–Littlewood, Schur's Q, Jack, zonal and Schur) are special cases of q-deformed symmetric functions. For $q=1$ and $s=0$, $P_\lambda^q(t)$ reduces to Hall–Littlewood symmetric functions and for $s=t^\alpha$, $q=1$ we get Jack symmetric functions, where α is an arbitrary parameter. For $q=1$ and $s=t$, $P_\lambda^q(s,t)$ reduces to S functions. We can also have q-deformations of symmetric functions by setting $q\neq 0,\pm 1$ any arbitrary complex number. For example, $P_\lambda^q(0,t)$ or simply $P_\lambda^q(t)$ is the q-deformation of the Hall–Littlewood symmetric function which is our major concern here.

Thus the scalar product $\langle,\rangle_{(t)}^{(q)}$ over $Q_q(t)$ is given by

$$\langle p_\lambda, p_\mu\rangle_{(t)}^{(q)} = \delta_{\lambda\mu}z_\lambda^q(t) \tag{8}$$

where

$$z_\lambda^q(t) = \prod_i [i]_q^{m_i}[m_i]_q!\prod_j^{l(\lambda)}\left(1-t^{[\lambda_j]_q}\right)^{-1}. \tag{9}$$

2.1. Duality and orthogonality

Let us introduce another symmetric function $Q^q_\lambda(t)$ related to $P^q_\lambda(t)$ by a scalar $b^q_\lambda(t)$ as follows

$$Q^q_\lambda(t) = b^q_\lambda(t) P^q_\lambda(t) \tag{10}$$

where

$$b^q_\lambda(t) = \prod_{i\geqslant 1} \phi^q_{m_i(\lambda)}(t) \qquad \phi^q_n(t) = \prod_{j\geqslant 1}^{n} \left(1 - t^{[j]_q}\right) \tag{11}$$

and m_i is the number of occurrences of i in λ. Then

$$\langle P^q_\lambda(t), Q^q_\mu(t)\rangle = \delta_{\lambda\mu}$$

i.e. $P^q_\lambda(t), Q^q_\lambda(t)$ are dual bases of Λ^q_Q for the scalar product $\langle\ ,\ \rangle$. It is easy to see that

$$\begin{aligned} Q^q_\lambda(t) &= \prod_{i<j} \left\{\frac{1-\delta_{ij}}{1-t\delta_{ij}}\right\}_q q^q_\lambda(t) \\ &= \prod_{i<j} \left(1 + (t^{[1]_q} - 1)\delta_{ij} + (t^{[2]_q} - t^{[1]_q})\delta^2_{ij} + \cdots\right) q^q_\lambda(t) \end{aligned} \tag{12}$$

where $q^q_\lambda(t)$ are the projection of $Q^q_\lambda(t)$ defined as

$$\prod_i \left\{\frac{1 - tx_i y}{1 - x_i y}\right\}_q = \sum_{r=0}^{\infty} q^q_r(x;t) y^r$$

and

$$q^q_\lambda(x;t) = \prod_i q^q_{\lambda_i}(x;t)$$

where y is an arbitrary parameter.

2.2. Recurrence relations of Q functions

The q-analogue of the recurrence relations obeyed by the Schur Q functions $Q_\lambda(-1)$ as given in [11] can be defined as

$$\begin{aligned} Q^q_{\lambda_1\lambda_2\cdots\lambda_l} = {} & Q^q_{\lambda_1\lambda_2} Q^q_{\lambda_3\lambda_4\cdots\lambda_l} - Q^q_{\lambda_1\lambda_3} Q^q_{\lambda_2\lambda_4\cdots\lambda_l} \\ & + \cdots + Q^q_{\lambda_1\lambda_l} Q^q_{\lambda_2\lambda_3\cdots\lambda_{l-1}} \qquad (l \text{ even}) \end{aligned}$$

and

$$Q^q_{\lambda_1\lambda_2\cdots\lambda_l} = q^q_{\lambda_1} Q^q_{\lambda_2\lambda_3\cdots\lambda_l} - q^q_{\lambda_2} Q^q_{\lambda_1\lambda_3\cdots\lambda_l} + \cdots + q^q_{\lambda_l} Q^q_{\lambda_2\lambda_3\cdots\lambda_{l-1}} \qquad (l \text{ odd})$$

and

$$Q^q_{\lambda_1\lambda_2} = q^q_{\lambda_1} q^q_{\lambda_2} - 2q^q_{\lambda_1+1} q^q_{\lambda_2-1} + \cdots + \left((-1)^{[\lambda_2]_q} + (-1)^{[\lambda_2-1]_q}\right) q^q_{\lambda_1+\lambda_2}.$$

The last relation is directly derived from equation (13). Also for $q^q_0 = 1$ and $q^q_{-s} = 0$ we have

$$Q^q_{-\lambda_r\lambda_r} = \left((-1)^{[\lambda_r]_q} + (-1)^{[\lambda_r-1]_q}\right) \qquad \text{and} \qquad Q^q_{\lambda_r,-\lambda_r} = 0.$$

3. q-analogue of the symmetric group S_n

The q-deformation of the symmetric functions leads to the q-analogue of the characters of S_n.

The connection between the ordinary characters of S_n and S functions can be given as

$$s_\lambda = \sum_\mu z_\mu^{-1} \chi_\mu^\lambda p_\mu \tag{13}$$

where χ_μ^λ is the character of the irrep $\{\lambda\}$ for the class $\{\mu\}$ and p_μ are power sum symmetric functions.

The spin characters are related to Schur's Q functions as follows

$$Q_\lambda = 2^{[(l(\lambda)+l(\nu)+1)/2]} \sum_\nu z_\nu^{-1} \zeta_\nu^{[\Delta;\lambda]} p_\nu \tag{14}$$

where $\zeta_\nu^{[\Delta;\lambda]}$ is the spin character for the class ν of odd cycles only and $[x]$ means the integer part of x.

We observe that for $s = t$, $P_\lambda^q(s,t)$ reduces to the q-deformed Schur function s_λ^q and for $s = 0$ & $t = -1$, $P_\lambda^q(s,t)$ reduces to the q-deformed Schur's Q function. Hence we can make a q-analogue of the equations (13) and (14) as follows

$$s_\lambda^q = \sum_\mu (z_\mu^q)^{-1} \chi_\mu^\lambda(q) p_\mu \tag{15}$$

and

$$Q_\lambda^q = 2^{[(l(\lambda)+l(\nu)+1)/2]} \sum_\nu (z_\nu^q)^{-1} \zeta_\nu^{[\Delta;\lambda]}(q) p_\nu . \tag{16}$$

3.1. *q-deformed spin characters*

In an earlier paper [13] we had presented the q-deformed ordinary characters of the symmetric group. The spin characters of S_n are normally calculated by using the recurrence relations of the Q functions along with (14) [11]. In this section we will use equation (16) and the q-analogue of the recurrence relations for the explicit calculations of the q-deformed spin characters.

Algorithm 1.

(i) Using (12), expand Q_λ^q in terms of q_r^q.

(ii) Write each q_r^q as

$$q_r^q = \sum_\rho (z_\rho^q)^{-1} 2^{l(\rho)} p_\rho$$

where ρ is a partition of r.

(iii) Equate this to expression (16) for Q_λ^q.

(iv) $\zeta_\nu^{\Delta;\lambda}(q)$ can be calculated by comparing the coefficients of p_ν on both sides of the equation (16).

Using this algorithm and (1) we give the q-deformed spin characters of S_4 in table 1.

It is important to note that the basic spin characters are independent of q.

Table 1. q-dependent spin characters of S_4.

	1^4	21^2	2^2	31	4
$[\Delta;0]_+$	2	0	0	1	$\sqrt{2}$
$[\Delta;0]_-$	2	0	0	-1	$-\sqrt{2}$
$[\Delta;1]$	$2(q+q^2+q^3-1)$	0	0	-1	0

4. q-analogue of vertex operators

Jing [5] has shown a relationship between vertex operators with a parameter t and the symmetric group S_n and its double covering group Γ_n. The parameter t plays a similar role in the description of vertex operators to one it plays in the theory of symmetric functions explained in the previous section, i.e. the vertex operators with $t=0$ correspond to S functions and those with $t=-1$ correspond to Schur's Q functions. Here we shall give a q-analogue of vertex operators and will show a 1:1 correspondence between the space of q-deformed vertex operators V_q and the ring of q-deformed symmetric functions $Q_q(t)$. The proofs given in this section will follow those in [6].

Vertex operators are defined with the help of infinite-dimensional Heisenberg algebras.

We shall define a q-analogue of a Heisenberg algebra $\mathcal{H}$ as

Definition 1. The q-Heisenberg algebra $\mathcal{H}_q$ is generated by a and ζ_n, $n \in \mathcal{Z}/0$, and satisfies the following relations

$$[\zeta_m, \zeta_n]_q = \frac{[m]_q}{1-t^{[m]_q}}\delta_{m+n,0}\,a \qquad [\zeta_m, a]_q = 0 \tag{17}$$

where t is a parameter.

As usual $S(\mathcal{H}_q^-)$ is the symmetric algebra generated by ζ_{-n}, $n \in \mathbb{N}$. ζ_{-n} is regarded as a multiplication operator and ζ_n as an annihilation operator on $S(\mathcal{H}_q^-)$. As an example,

$$\zeta_n\zeta_{-n}\cdot 1 = \frac{[n]_q}{1-t^{[n]_q}} \qquad n \in \mathbb{N}$$

where a is considered as an identity operator.

Now we can define the q-analogue of a simplified form of vertex operators on the space $S(\mathcal{H}_q^-)$ as follows

$$\begin{aligned} V(x) &= \exp\left\{\sum_{n\geqslant 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}x^n\right\}\exp\left\{-\sum_{n\geqslant 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_n x^{-n}\right\} \\ &= \sum_{n\in\mathbb{Z}} V_n x^{-n} \end{aligned} \tag{18}$$

$$\begin{aligned} V^*(x) &= \exp\left\{-\sum_{n\geqslant 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}x^n\right\}\exp\left\{\sum_{n\geqslant 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_n x^{-n}\right\} \\ &= \sum_{n\in\mathbb{Z}} V_n^* x^n. \end{aligned} \tag{19}$$

We define a Hermitian structure $\langle , \rangle$ in the space $S(\mathcal{H}_q^-)$

$$\langle \zeta_{-n}, \zeta_{-n} \rangle = \frac{[n]_q}{1 - t^{[n]_q}}$$

or, in general,

$$\langle \zeta_{-\lambda}, \zeta_{-\mu} \rangle = z_\lambda^q(t) \delta_{\lambda\mu}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ are partitions.

A polynomial function in ζ_{-n} can be defined as follows

$$\exp\left(t^{[n]_q} [n]_q \zeta_{-n} x^n\right) = \sum_{n \geqslant 0} R_n^q(t) x^n. \tag{20}$$

Hence

$$R_n^q(t) = \sum_{|\lambda| = n} \left(z_\lambda^q(t)\right)^{-1} \zeta_{-\lambda} \qquad \zeta_{-\lambda} = \zeta_{-\lambda_1} \zeta_{-\lambda_2} \cdots \zeta_{-\lambda_l}. \tag{21}$$

The normal ordering product is used when the annihilation operator has to be moved to the right of the product [3], as shown here.

$$: V(x) V(y) := \exp\left\{ \sum_{n \geqslant 1} \frac{1 - t^{[n]_q}}{[n]_q} \zeta_{-n} (x^n + y^n) \right\}$$
$$\times \exp\left\{ - \sum_{n \geqslant 1} \frac{1 - t^{[n]_q}}{[n]_q} \zeta_n (x^{-n} + y^{-n}) \right\}$$

and

$$V(x) V(y) =: V(x) V(y) : \left\{ \frac{x - y}{x - ty} \right\}_q$$

where the subscript q indicates that the factor $\{(x - y/x - ty)\}_q$ is a formal series in y/x with the powers of t being q-numbers.

Using the q-analogue of Young raising operators we give a q-analogue of Jing's proposition (2.17) [6] as follows.

Theorem 1. For a partition $\lambda = (\lambda_1 \lambda_2 \ldots \lambda_l)$ the element $V_{-\lambda} \cdot 1$ can be expressed as

$$V_{-\lambda} \cdot 1 = \prod_{i<j} \left\{ \frac{1 - \delta_{ij}}{1 - t\delta_{ij}} \right\}_q R_\lambda^q(t)$$

where δ_{ij} is Young's raising operator whose action is defined as

$$\delta_{ij} R^q_{(\lambda_1 \ldots \lambda_i \ldots \lambda_j \ldots)} = R^q_{(\lambda_1 \ldots \lambda_i + 1 \ldots \lambda_j - 1 \ldots)}$$

and the subscript q in $\{\ \}_q$ means the powers of t are q-numbers.

Proof. The action of the components of the vertex operators $V(x)$ as defined in (18) can be shown to be

$$V_{-n}.1 = \frac{1}{2\pi i}\int_c \exp\left(\sum_{m\geqslant 1}\frac{1-t^{[m]_q}}{[m]_q}\zeta_{-m}x^m\right)X^{-n}\frac{\mathrm{d}x}{x}$$

where the subscript c is for the contour integral.

Then it is easy to see the trivial result

$$V_{-n}\cdot 1 = R^q_n(t).$$

For the rest, let us use the contour integral approach. For any partition $\lambda = \lambda_1 \dots \lambda_l$,

$$V_{-\lambda}\cdot 1 = \underbrace{\int\cdots\int}_{l} V(x_1)\dots V(x_l)\cdot 1 x^{-\lambda}\frac{\mathrm{d}x_1}{x_1}\cdots\frac{\mathrm{d}x_l}{x_l}$$

$$= \frac{1}{(2\pi \mathrm{i})^l}\int \exp\left(\sum_{i=1,n\geqslant 1}^{l}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}x_i^n\right)\prod_{1\leqslant i<j\leqslant l}\left\{\frac{x_i-x_j}{x_i-tx_j}\right\}_q x^{-\lambda}\frac{\mathrm{d}x}{x}$$

where the term $\{x_i - x_j/x_i - tx_j\}_q$, comes from the normal ordering of the creation and annihilation operators. Using the definition of $R^q_n(t)$, we can write the following.

$$V_{-\lambda}\cdot 1 = \frac{1}{(2\pi \mathrm{i})^l}\int\sum_{n\in N^l}R^q_n(t)\prod_{i<j}\left\{\frac{x_i-x_j}{x_i-tx_j}\right\}_q x^{-\lambda+n}\frac{\mathrm{d}x}{x}.$$

Expanding the formal series $\{x_i - x_j/x_i - tx_j\}_q$ for $i = 1$ we get

$$V_{-\lambda}\cdot 1 = \frac{1}{(2\pi \mathrm{i})^l}\int\sum_{n\in N^l}R^q_n(t)\prod_{j=2}^{l}\left(1+(t^{[1]_q}-1)\frac{x_j}{x_1}+(t^{[2]_q}-t^{[1]_q})\frac{x_j^2}{x_1^2}+\cdots\right)$$

$$\times\prod_{2\leqslant i<j\leqslant l}\left\{\frac{x_i-x_j}{x_i-tx_j}\right\}_q x^{-\lambda+n}\frac{\mathrm{d}x}{x}$$

$$= \frac{1}{(2\pi \mathrm{i})^{l-1}}\prod_{j=2}^{l}\left(1+(t^{[1]_q}-1)\delta_{1j}+(t^{[2]_q}-t^{[1]_q})\delta_{1j}^2+\cdots\right)$$

$$\times\int\sum_{\tilde{n}}R^q_{\lambda_1}(t)R^q_{\tilde{n}}(t)$$

$$\times\prod_{2\leqslant i<j\leqslant l}\left\{\frac{x_i-x_j}{x_i-tx_j}\right\}_q \tilde{x}^{-\tilde{\lambda}+\tilde{n}}\frac{\mathrm{d}\tilde{x}}{\tilde{x}}$$

$$= \prod_{i<j}\left\{\frac{1-\delta_{ij}}{1-t\delta_{ij}}\right\}_q R^q_\lambda(t)$$

where $\tilde{\lambda} = \lambda_2, \lambda_3, \dots \lambda_l$, $\tilde{x} = x_2 \dots x_l$, $\mathrm{d}x/x = \mathrm{d}x_1/x_1 \dots \mathrm{d}x_l/x_l$ and $R^q_\lambda(t) = R^q_{\lambda_1}(t)R^q_{\lambda_2}(t)\dots R^q_{\lambda_l}(t)$. The orthogonality of the q-analogue of vertex operators can be described by the following theorem.

Theorem 2. For two partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$

$$\langle V_{-\lambda}.1, V_{-\mu}.1\rangle_q = b_\lambda^q(t)\delta_{\lambda\mu} \tag{22}$$

where

$$b_\lambda^q(t) = \prod_{i\geqslant 1} \phi^q_{m_i(\lambda)}(t) \qquad \phi_n^q(t) = \prod_{j\geqslant 1}^{n} \left(1 - t^{[j]_q}\right) \tag{23}$$

and m_i is the number of occurrences of i in λ.

In order to prove this we will give q-analogues of some of Jing's results [6].

Lemma 1. For $m, n \in N$, we have

$$V^*_{-n} V_{-m} \cdot 1 = \delta_{m,n}(1 - t^{[1]_q}).$$

The proof of this lemma is straightforward using the properties of the components of vertex operators.

Proposition 1. Let $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ and $\tilde{\lambda} = (1^{m_1 - 1}, 2^{m_2}, \ldots)$ then we have

$$V^*_{-n} V_{-\lambda} \cdot 1 = \delta_{n,\lambda_1} \left(1 - t^{[m_1]_q}\right) V_{-\tilde{\lambda}}.1$$

The previous lemma and the inductive assumptions prove this proposition.

Now the orthogonality of the q-deformed vertex operators can be proved as follows.

For two partitions λ and μ, such that $|\lambda| = |\mu|$ we have

$$\begin{aligned}\langle V_{-\lambda}.1, V_{-\mu}.1\rangle_q &= V_{-\tilde{\lambda}}.1, V^*_{-\lambda_1} V_{-\mu}.1\rangle_q \\ &= \langle V_{-\tilde{\lambda}}.1, \delta_{\lambda_1,\mu_1}\left(1 - t^{[m_1(\mu)]_q}\right) V_{-\tilde{\mu}}.1\rangle_q \\ &= \delta_{\lambda_1,\mu_1}\left(1 - t^{[m_1(\mu)]_q}\right)\langle V_{-\tilde{\lambda}}.1, V_{-\tilde{\mu}}.1\rangle_q.\end{aligned}$$

By repeating this we get

$$\langle V_{-\lambda}.1, V_{-\mu}.1\rangle_q = b_\lambda^q(t)\delta_{\lambda\mu}$$

which is the desired result. □

Comparing the inner product $\langle \zeta_{-\lambda}, \zeta_{-\mu}\rangle = z_\lambda^q(t)\delta_{\lambda\mu}$, and (8), we can define a mapping from V_q to $\Lambda \otimes_Z Q_q(t)$ as follows.

Definition 2. The mapping $\rho : \mathcal{V}_q \to \Lambda \otimes_Z \mathcal{Q}_q(t)$ for a partition $\lambda = (1^{m_1}2^{m_2}...l^{m_l})$ is given by

$$\rho(\zeta_{-\lambda}) = \rho(\zeta_{-1}^{m_1}\zeta_{-2}^{m_2}...\zeta_{-l}^{m_l}) = p_1^{m_1}p_2^{m_2}...p_l^{m_l} = p_\lambda.$$

This immediately gives

$$\rho(V_{-\lambda}\cdot 1) = \rho(V_{-\lambda_1}V_{-\lambda_2}...V_{-\lambda_l}\cdot 1) = \prod_{i<j}\left\{\frac{1-\delta_{ij}}{1-t\delta_{ij}}\right\}_q q_\lambda^q. \tag{24}$$

Comparing (24) and the identity

$$\langle Q_\lambda^q(t), Q_\mu^q(t)\rangle = b_\lambda^q(t)\delta_{\lambda\mu}$$

we conclude that for a general value of t the map $\rho : \mathcal{V}_q \to \Lambda \otimes_Z \mathcal{Q}_q(t)$ takes the form

$$\rho(V_{-\lambda_1}V_{-\lambda_2}...V_{-\lambda_l}(t).1) = Q_\lambda^q(t). \tag{25}$$

The specializations $t = 0, -1$ give the following results

$$\rho(V_{-\lambda_1}V_{-\lambda_2}.V_{-\lambda_l}(-1).1 = Q_\lambda^q(-1) \tag{26}$$

and

$$\rho(V_{-\lambda_1}V_{-\lambda_2}...V_{-\lambda_l}(0).1) = s_\lambda^q \tag{27}$$

where $Q_\lambda^q(-1)$ are the q-deformed Schur Q functions and s_λ^q are the q-deformed S functions.

5. Construction of untwisted q-vertex operators

In previous sections we have worked out the q-analogue of the vertex operators. On the basis of (18) and (19) we define the untwisted q-vertex operators in normal ordered form as follows.

$$\begin{aligned} U(z) &= \exp\left\{\sum_{n\geqslant 1}\frac{1-t^{[n]}}{[n]}\zeta_{-n}z^n\right\}\exp\left\{-\sum_{n\geqslant 1}\frac{1-t^{[n]}}{[n]}\zeta_n z^{-n}\right\}e^{\zeta}z^{\zeta_{(0)}+1} \\ &= \sum_{n\in\mathbb{Z}} U_n z^{-n} \\ &=: U(z): \end{aligned} \tag{28}$$

and

$$\begin{aligned} U^*(z) &= \exp\left\{-\sum_{n\geqslant 1}\frac{1-t^{[n]}}{[n]}\zeta_{-n}z^n\right\}\exp\left\{\sum_{n\geqslant 1}\frac{1-t^{[n]}}{[n]}\zeta_n z^{-n}\right\}e^{\zeta}z^{-(\zeta_{(0)}+1)} \\ &= \sum_{n\in\mathbb{Z}} U_n^* z^n \\ &=: U^*(z): \end{aligned} \tag{29}$$

where z is a non-zero complex number and the action of $\zeta_{(0)}$ is defined as

$$\zeta_{(0)}\mathrm{e}^{\eta} = \langle \eta, \zeta \rangle \mathrm{e}^{\eta} \qquad \eta, \zeta \in S(\mathcal{H}_q^-).$$

The factors $\mathrm{e}^{\zeta} z^{-(\zeta_{(0)}+1)}$ and $\mathrm{e}^{\zeta} z^{\zeta_{(0)}+1}$ arise from the commutation of annihilation operators as they are transferred to the right in accordance with normal ordering. For $t \to 0$ and $q \to 1$ these expressions take a similar form to the vertex operators used in dual resonance theory [4].

There is another way of developing the untwisted q-vertex operators. Consider a finitely generated free Abelian group L and define a non-singular symmetric $\mathcal{Z}$-bilinear form $\langle , \rangle$ on L such that

$$\langle \zeta, \zeta \rangle \in 2\mathcal{Z} \qquad \text{for} \qquad \zeta \in L.$$

Define the function

$$C : L \times L \to \mathcal{F}$$
$$(\zeta, \eta) \mapsto (-1)^{\langle \zeta, \eta \rangle} \omega^{\langle m\zeta, \eta \rangle} = \prod (-\omega^m)^{\langle \zeta, \eta \rangle}$$

where ω is the kth primitive root of unity and $m \in \mathcal{Z}/k\mathcal{Z}$. Then the commutator map C is bilinear into the Abelian group $\mathcal{F}$ such that

$$\begin{aligned} C(\zeta + \eta, \theta) &= C(\zeta, \theta) C(\eta, \theta) \\ C(\zeta, \eta + \theta) &= C(\zeta, \eta) C(\zeta, \theta) \end{aligned} \tag{30}$$

and

$$C(\zeta, \zeta) = 1 \tag{31}$$

for $\zeta, \eta, \theta \in L$.

Let $\omega_0 = (-1)^k \omega$. In view of equations (30) and (31) there is a unique central extension

$$1 \to \langle \omega_0 \rangle \to \hat{L} \xrightarrow{-} L \to 1 \tag{32}$$

of L by the cyclic group generated by ω_0 with commutator map C such that

$$aba^{-1}b^{-1} = C(\bar{a}, \bar{b}) \qquad \text{for } a, b \in \hat{L}.$$

We fix $a \in \hat{L}$ such that $\bar{a} = \zeta$. This construction gives us the following form of the q-vertex operators.

$$\begin{aligned} \mathcal{X}(z) &= \exp\left\{ \sum_{n \geqslant 1} \frac{1 - t^{[n]_q}}{[n]_q} \zeta_{-n} z^n \right\} \exp\left\{ -\sum_{n \geqslant 1} \frac{1 - t^{[n]_q}}{[n]_q} \zeta_n z^{-n} \right\} a z^{\zeta_{(0)}+1} \\ &= \sum_{n \in \mathbf{Z}} \mathcal{X}_n z^n \end{aligned} \tag{33}$$

where the $a \in \hat{L}$. For a special case of

$$\langle \omega_0 \rangle \equiv \langle \pm 1 \rangle$$

we get

$$aba^{-1}b^{-1} = (-1)^{\langle \bar{a}, \bar{b} \rangle} \qquad \text{for } a, b \in \hat{L}$$

and the untwisted q-vertex operators take the following form.

$$\mathcal{X}^{\pm}(z) = \exp\left\{ \pm \sum_{n \geqslant 1} \frac{1 - t^{[n]_q}}{[n]_q} \zeta_{-n} z^n \right\} \exp\left\{ \mp \sum_{n \geqslant 1} \frac{1 - t^{[n]_q}}{[n]_q} \zeta_n z^{-n} \right\} a^{\pm 1} z^{\pm \zeta_{(0)} + 1}$$
$$= \sum_{n \in \mathbb{Z}} \mathcal{X}_n^{\pm} z^n. \tag{34}$$

For $t = 0$ this expression is similar to the one given by Frenkel and Jing [2] in the description of quantum affine algebras, except that they have used a different definition of q-number. Equation (34) is the most general expression for the vertex operators of *untwisted* type. With the various specializations of q and t one can derive all the vertex operators discussed earlier.

6. Construction of twisted q-vertex operators

Twisted vertex operators are now obtained by the action of an automorphism of a certain group M defined below. Closely following the terminology and notation used in [8] and using the results of the previous section, we define the following.

(i) M is a finitely generated free Abelian group.

(ii) $\langle , \rangle$ is a non-singular symmetric $\mathcal{Z}$-bilinear form on M such that

$$\langle \zeta, \zeta \rangle \in 2\mathcal{Z} \qquad \text{for } \zeta \in M.$$

(iii) σ is an automorphism of M such that

$$\langle \sigma\zeta, \sigma\eta \rangle = \langle \zeta, \eta \rangle \qquad \text{for } \zeta, \eta \in M.$$

(iv) m is a positive integer such that $\sigma^m = 1$.

(v)

$$\left\langle \sum_{p \in \mathcal{Z}/m\mathcal{Z}} \sigma^p \zeta, \zeta \right\rangle \in 2\mathcal{Z} \qquad \text{for } \zeta \in M.$$

Considering the action of the automorphism σ we redefine the commutator map C as follows.

$$C : M \times M \to \mathcal{F}$$

$$(\zeta, \eta) \mapsto (-1)^{\langle \sum_{p \in \mathcal{Z}/m\mathcal{Z}} \sigma^p \zeta, \eta \rangle} \omega^{\langle \sum_{p \in \mathcal{Z}/m\mathcal{Z}} p\sigma^p \zeta, \eta \rangle} \prod_{p \in \mathcal{Z}/m\mathcal{Z}} (-\omega^p)^{\langle \sigma^p \zeta, \eta \rangle}.$$

Along with equations (30) and (31) we include the following

$$C(\sigma\zeta,\sigma\eta)=C(\zeta,\eta) \qquad \text{for } \zeta,\eta\in M \tag{35}$$

and

$$C(\zeta,\eta)=C(\eta,\zeta)^{-1} \qquad \text{for } \zeta,\eta\in M. \tag{36}$$

Then the central extension of M by the cyclic group generated by ω_0 with the commutator map C is

$$1\to\langle\omega_0\rangle\to\hat{M}\xrightarrow{-}M\to 1 \tag{37}$$

such that

$$aba^{-1}b^{-1}=C(\bar{a},\bar{b}) \qquad \text{for } a,b\in\hat{M}.$$

The automorphism σ can be *extended* to an automorphism $\hat{\sigma}$ of the extension $\hat{M}$ of M such that

$$(\hat{\sigma}a)^{-}=\sigma\bar{a} \qquad \forall\ a\in\hat{M}$$

and

$$\hat{\sigma}a=a\omega^{-\sum_{p\in\mathbb{Z}/m\mathbb{Z}}\sigma^{p}\bar{a}(0)-\langle\sigma^{p}\bar{a},\bar{a}\rangle/2}.$$

Now the twisted q-vertex operators can be defined as

$$\mathcal{X}(z)=\mathcal{E}_{-}(\zeta,z)\mathcal{E}_{+}(\zeta,z)az^{-\sum_{p\in\mathbb{Z}/m\mathbb{Z}}\sigma^{p}\bar{a}(0)-\langle\sigma^{p}\bar{a},\bar{a}\rangle/2} \tag{38}$$

where

$$\mathcal{E}_{\pm}=\exp\left\{\sum_{\pm n\geqslant 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}z^{n}\right\}.$$

Again for the special case of

$$\langle\omega_0\rangle\equiv\langle\pm 1\rangle$$

we get

$$aba^{-1}b^{-1}=(-1)^{\langle\bar{a},\bar{b}\rangle} \qquad \text{for } a,b\in\hat{M}$$

and the twisted q-vertex operators take the form

$$\begin{aligned}\mathcal{X}^{\pm}(z)&=\mathcal{E}_{-}^{\pm}(\zeta,z)\mathcal{E}_{+}^{\pm}(\zeta,z)a^{\pm 1}z^{\pm\sum_{p\in\mathbb{Z}/m\mathbb{Z}}\sigma^{p}\bar{a}(0)-\langle\sigma^{p}\bar{a}\bar{a}\rangle/2}\\&=\sum_{n\in\mathbb{Z}}\mathcal{X}_{n}^{\pm}z^{n}\end{aligned} \tag{39}$$

where

$$\mathcal{E}_{-}^{\pm}=\exp\left\{\pm\sum_{n\geqslant 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}z^{n}\right\}$$

and

$$\mathcal{E}_{+}^{\pm}=\exp\left\{\mp\sum_{n\geqslant 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{n}z^{n}\right\}.$$

For the specialization $t=0$ equation (39) gives a similar result to the one reported in [7] except that the definition of q-number is different. Also for $q=1$ and $t=0$ this result is similar to the case studied by Lepowsky [8]. We find that the expression (39) is a very general form of vertex operators. Various specializations of t, q and σ give the desired results. For example, in the case of identity automorphism $\sigma=1$ we get the *untwisted* q-vertex operators and equation (39) reduces to (34).

7. Conclusion

A q-analogue of the Heisenberg algebra is defined. This leads to the construction of q-vertex operators with a parameter t similar to the theory of symmetric functions. An isomorphism from the space of q-vertex operators to the ring Λ^q_Q of the q-deformed vertex operators is defined explicitly. This isomorphism is valid for a general value of t and as well as the specialized values such that $t = 0$ and $t = -1$ in which case we get S functions and Schur Q functions. Using these results a very simple technique for the construction of *twisted* and *untwisted* q-vertex operators is developed. This approach is more simple and straightforward than any other technique. The final result is a very general form of the vertex operators and by the specializations of various parameters, the results can be verified.

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